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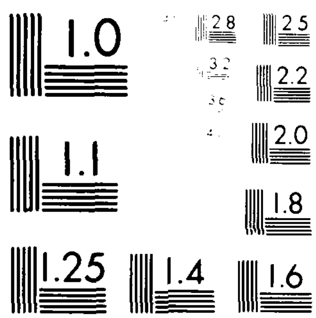

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Period: 30 September 1980 through 29 September 1981

Title of Research: Studies in Numerical Algebra with Applications

Principal Investigators: M. Marcus
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Research either appeared or in press during the period 1 Oct 80 - 30 Sep 81, covered the following topics: (1) generalizations of the Cauchy-Schwarz inequality to determinants; (2) the structure of the set of complex numbers $(Ax_1, Y_1) + \ldots + (Ax_r, Y_r)$ where the inner products (x_i, x_j), (Y_i, Y_j) and (x_i, Y_j) are stipulated, $i = 1, \ldots, r$; (3) the positive definite property of the A-numerical radius defined by Goldberg and Straus; (4) conditions for two decomposable tensors in a symmetry class to be equal or for a decomposable tensor to be 0; (5) the geometry of the set of numbers (CONTINUED)

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ITEM #20, CONTINUED: (Ax, Y) in which x and Y are unit vectors with fixed inner product. This report summarizes the current research of M. Marcus sponsored by the Air Force Office of Scientific Research covering the general topic of the geometry of the quadratic and bilinear numerical ranges.

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ABSTRACT

MARVIN MARCUS: Eigenvalue localization; quadratic and bilinear numerical ranges; symmetry classes of tensors

Research ~~either~~ appeared or in press during the period October 1, 1980 - September 30, 1981 covered the following topics: (1) generalizations of the Cauchy-Schwarz inequality to determinants; (2) the structure of the set of complex numbers $(Ax_1, y_1) + \cdots + (Ax_r, y_r)$ where the inner products (x_i, x_j), (y_i, y_j) and (x_i, y_j) are stipulated, $i = 1, \ldots, r$; (3) the positive definite property of the A-numerical radius defined by Goldberg and Straus; (4) conditions for two decomposable tensors in a symmetry class to be equal or for a decomposable tensor to be 0; (5) the geometry of the set of numbers (Ax, y) in which x and y are unit vectors with fixed inner product.

This report summarizes the current research of M. Marcus sponsored by the Air Force Office of Scientific Research covering the general topic of the geometry of the quadratic and bilinear numerical ranges.

1. <u>Generalizations of the Cauchy-Schwarz inequality to determinants.</u>

In answer to a research problem posed by A. Abian [1], we were led to consider the inequality (1) defined below.

Let A, B, P, Q be n-square complex matrices and for $1 \le m < n$, let U and V be $n \times m$ complex matrices. We

examine the relations which must exist between A, B, P, and Q so that

$$\det(U^*AV)\det(V^*BU) \leq \det(U^*PU)\det(V^*QV) \qquad (1)$$

holds for all U and V. If $m = 1$ and A, B, P, Q are taken to be the n-square identity matrix, then (1) reduces to the Cauchy-Schwarz inequality

$$|(u,v)|^2 \leq (u,u)(v,v),$$

where u and v are arbitrary $n \times 1$ column vectors.

If H is hermitian positive definite (negative definite), we write $H > 0$ $(H < 0)$. If H and K are definite of the same sign (opposite signs), we write $HK > 0$ $(HK < 0)$. The principal result generalizing the Cauchy-Schwarz inequality is contained in the following theorem [13].

Theorem. Assume $n \geq 3$, $1 \leq m < n$, and A, B, P, Q are nonsingular complex matrices. Then (1) holds for all $n \times m$ rectangular matrices U and V if and only if

(i) $P = \alpha H$, $Q = \beta K$, $(\alpha\beta)^m = \varepsilon = \pm 1$, H and K are definite hermitian and

(ii) $A^* = \omega B$, $\omega \in C$, $\omega^m = \lambda \in R$, so that (1) reads

$$\lambda|\det(V^*BU)|^2 \leq \varepsilon\det(U^*HU)\det(V^*KV) \qquad (2)$$

and if $\xi_1, \ldots, \xi_n$ are the eigenvalues of $H^{-1}B^*K^{-1}B$ and

$$\xi = \max_{\alpha \in Q_{m,n}} \frac{\lambda}{\varepsilon} \prod_{i=1}^{m} \xi_{\alpha(i)}$$

then:

(iii) if $\varepsilon = 1$, $\lambda > 0$, then either m is odd, $HK > 0$, and $\xi \leqq 1$, or m is even, HK can have either sign, and $\xi \leqq 1$; or

(iv) if $\varepsilon = 1$, $\lambda < 0$, then either m is odd and $HK > 0$, or m is even and HK can have either sign; or

(v) if $\varepsilon = -1$, $\lambda > 0$, then m is odd, $HK < 0$, and $\xi \leq 1$; or

(vi) if $\varepsilon = -1$, $\lambda < 0$, then m is odd and $HK < 0$.

We include a table of conditions (iii) - (vi):

ε	λ	m	H	K	ξ
1	+	odd	$\pm$	$\pm$	≤ 1
1	+	even			$\leqq 1$
1	-	odd	$\pm$	$\pm$	
1	-	even			
-1	+	odd	$\pm$		$\leqq 1$
-1	-	odd	$\pm$		

It should be noted that for n = 2, m = 1 the inequality (2) can hold with both H and K indefinite. Simply take

$$B = I_2, \quad H = K = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \lambda = \varepsilon = -1.$$

2. The G-bilinear range.

The númerical range of a linear operator $A:V \to V$, denoted by $W(A)$, is the image of the surface of the unit ball under the quadratic mapping

$$x \to (Ax,x). \tag{3}$$

That is,

$$W(A) = \{(Ax,x) \mid \|x\| = 1\}.$$

In 1918 and 1919 [2,3] Toeplitz and Hausdorff proved the celebrated result:

W(A) is a convex set in the plane.

Since that time a long sequence of proofs, generalizations and refinements of this result have appeared [4]. Perhaps the two most significant of these appeared in 1937 and 1951. The first of these was a paper by von Neumann [5] in which the values of

$$\mathrm{tr}(XAYB) \tag{4}$$

were examined for A and B fixed linear operators on V with X and Y varying over all unitary operators. von Neumann showed that the totality of values (4) is a disk centered at the origin whose radius is

$$\sum_{i=1}^{n} \alpha_i \beta_i \tag{5}$$

where $\alpha_1 \geqq \cdots \geqq \alpha_n$ and $\beta_1 \geqq \cdots \geqq \beta_n$ are the singular values of A and B respectively. The second result, due to Fan [6], dealt with values of

$$\sum_{i=1}^{r} (Ax_i , x_i) \tag{6}$$

in which $x_1,\ldots,x_r$ are arbitrary orthonormal (o.n.) vectors. Of course (6) is similar in form to (4). In fact, the set of values (6) is precisely the same as the set

$$W_r(A) = \{tr(PA) \mid P \text{ is an orthogonal projection of rank } r\}. \tag{7}$$

In his book [7] Halmos calls the set $W_r(A)$ the r^{th} higher numerical range. This set is also convex for any A, a fact first proved by Berger [8] for $r \geqq 1$. More recently, Goldberg and Straus [9] considered the C-numerical range of $A:V \to V$

$$W_C(A) = \{tr(CU^*AU) \mid U \text{ unitary}\} \tag{8}$$

where C is a fixed operator on V. It is not true that (8) is a convex set for any C, however if C is hermitian then $W_C(A)$ is convex for any A, a result originally due to Westwick [10] and quite recently reproved using algebraic methods by Poon [11],

The numerical range can be regarded as a special case of the q-<u>bilinear range</u>

$$W(A:q) = \{(Ax,y) \mid \|x\| = \|y\| = 1,\ (x,y) = q\}. \tag{9}$$

If $q = 1$, $W(A:q) = W(A)$. In 1961 Bush and Olkin [12] showed that if V is a Euclidean space over R, $q > 0$, and A is symmetric then $W(A:q)$ is the real interval

$$[((1+q)\lambda_n - (1-q)\lambda_1)/2\ ,\ ((1+q)\lambda_1 - (1-q)\lambda_n)/2], \tag{10}$$

where $\lambda_1 \geq \cdots \geq \lambda_n$ are the eigenvalues of A. In [9] the result (10) is generalized in various ways.

For $1 \leq r \leq n$ let G be an $r \times 3r$ complex matrix partitioned into three $r \times r$ submatrices:

$$G = [G_1 : G_2 : G_3]. \tag{11}$$

A set of 2r vectors $x_1,\ldots,x_r$, $y_1,\ldots,y_r$ are G-<u>vectors</u> if

$$[(x_i,x_j)] = G_1, \tag{12}$$

$$[(y_i, y_j)] = G_2, \tag{13}$$

and

$$[(x_i, y_j)] = G_3. \tag{14}$$

We shall assume henceforth that G_1 and G_2 are positive definite hermitian so that if $x_1, \ldots, x_r$ and $y_1, \ldots, y_r$ are G-vectors then $x_1, \ldots, x_r$ are linearly independent as are $y_1, \ldots, y_r$. In this paper we investigate the values of the function

$$\varphi(x;y) = \sum_{k=1}^{r} (Ax_k, y_k) \tag{15}$$

as $x_1, \ldots, x_r$ and $y_1, \ldots, y_r$ run over all sets of G-vectors. We define the G-<u>bilinear</u> <u>range</u> to be the set

$$W(A;G) = \{\varphi(x;y) | \ x_1, \ldots, x_r \text{ and } y_1, \ldots, y_r \text{ G-vectors}\}. \tag{16}$$

Of course the set $W(A;G)$ may be empty for certain choices of G. In fact we shall say that G is <u>admissable</u> if and only if there exist G-vectors. Henceforth we shall assume that G is admissable so that $W(A;G)$ is not empty. In Theorem 3 we will obtain some necessary and sufficient conditions for admissability.

The principal results appearing in [14] are

Theorem 1. <u>If</u> $1 \le r \le n$, $G_1G_2 = G_2G_1$, <u>and</u>

$$G_3 = G_1^{1/2} G_2^{1/2} \tag{17}$$

then for any A, W(A;G) is a convex set in the plane.

Theorem 2. If $G_3 \neq \lambda I_n$ and tr $(G_3) \neq 0$ then W(A;G) is the origin if and only if A = 0.

Theorem 2 is closely related to a result of Goldberg and Straus [9]. Note that if r < n then the conditions $G_3 \neq \lambda I_n$ automatically holds.

The conditions for admissability are:

Theorem 3. The r × 3r matrix $G = [G_1 : G_2 : G_3]$ is admissable for the n-dimensional unitary space V if and only if

a) the largest eigenvalue of

$$G_1^{-1} G_3 G_2^{-1} G_3^* \tag{18}$$

does not exceed 1 and

b) if (18) has 1 as an eigenvalue its multiplicity k satisfies

$$n - r \geq r - k.$$

3. Elementary proofs of the Goldberg-Straus theorem.

For a pair of n-square, complex matrices A and B, M. Goldberg and E.G. Straus [9] have defined the A-numerical radius of B as the quantity

$$r_A(B) = \max |\mathrm{tr}(AU^*BU)|,$$

where the maximum is taken over all unitary matrices U. In a later paper [15] they showed that r_A is a generalized matrix norm if and only if the conditions

$$A \text{ is nonscalar and } \operatorname{tr} A \neq 0 \tag{19}$$

hold. A generalized matrix norm is a scalar valued, homogeneous, subadditive, positive definite matrix function. The homogeneity and subadditivity properties are easily established, as is the necessity of the condition (19) from the positive definiteness of r_A. Thus the majority of section 2 in the previously cited paper is concerned with showing that condition (19) implies the positive definiteness of the A-numerical radius.

While examining the proof of this last statement in a seminar held during the summer of 1980 in which M. Goldberg participated, three separate proofs of this result were obtained by the present authors.

4. Symmetry classes of tensors.

In a paper by M. Marcus and J. Chollet [17], necessary and sufficient conditions were stated for two decomposable symmetrized tensors to be equal. There, the linear independence of the vectors forming such tensors was assumed. In [24] this assumption is dropped and more general requirements for equality are obtained. Conditions

for a decomposable symmetrized tensor to be zero are also given.

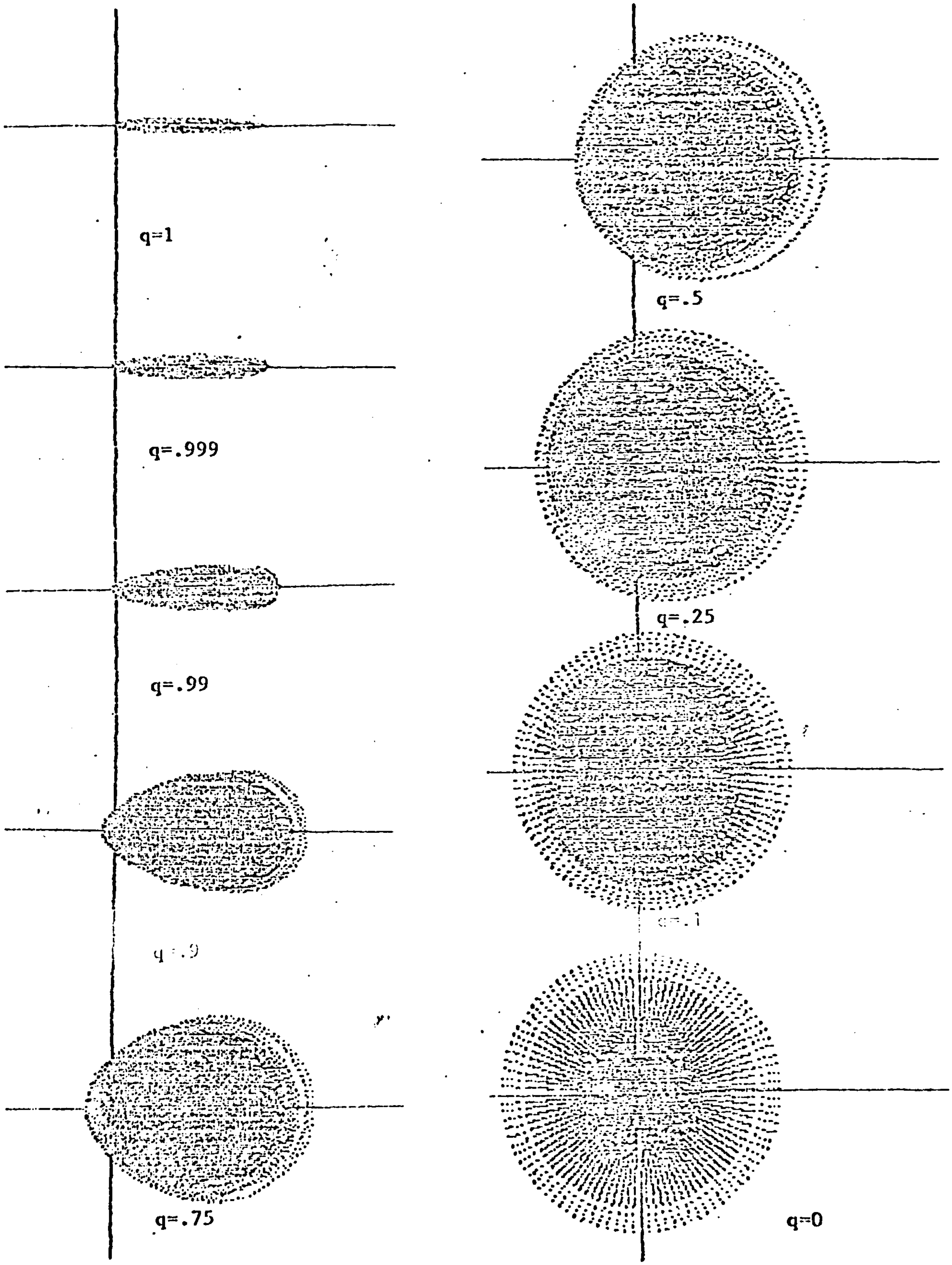
A brief history of the above problem is as follows. Before the paper of Dias da Silva [18], most results concerned symmetry classes whose associated characters were of degree one (see theorem 4.5, p.136, of [19]). R. Merris [20] did consider more general symmetrizers and higher degree characters but did not give necessary and sufficient conditions for equality. This was done by Dias da Silva in [18] and then with de Oliveira in [21]. A different approach was initiated in [17] and is continued here. Related results have appeared in [22] and [23].

5. The q-bilinear range

Numerical experiments are currently under way on the q-bilinear range

$$W(A:q) = \{(Ax,y) \mid (x,x) = (y,y) = 1,\ (x,y) = q\}.$$

It is conjectured that the set W(A:q) is convex for $0 < q < 1$. For $n = 2$, the following graphic outputs were obtained using an APPLE II-plus computer and the GRAPHTRIX plotting software by DATA TRANSFORMS, INC.



$$\begin{bmatrix} 1 & .1 \\ 0 & 0 \end{bmatrix}$$

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Marvin Marcus
Publications: October 1, 1980 to date

1. M. Marcus and K. Moore, A determinant formulation of the Cauchy-Schwarz inequality, Linear Algebra Appl. 36(1981), 111-127.

2. M. Marcus and J. Chollet, On the equality of decomposable symmetrized tensors, to appear, Linear and Multilinear Algebra.

3. M. Marcus and J. Chollet, The index of a symmetry class of tensors, to appear, Linear and Multilinear Algebra.

4. M. Marcus and M. Sandy, Three elementary proofs of the Goldberg-Straus theorem on numerical radii, to appear, Linear and Multilinear Algebra, 11(3), 1982.

5. M. Marcus, K. Kidman and M. Sandy, Bilinear ranges, in preparation.

6. Solution to Problem #1122 (with M. Newman), Mathematics Magazine, to appear.

Abstract

M. Newman: THE APPLICATIONS OF NUMBER THEORY TO COMPUTATION

Research completed in the last period includes the following:

(1) The preparation of programs using modular arithmetic for serial machines to find all rational solutions of a system of linear equations. (2) The preparation of programs using multiprecision integral arithmetic for serial machines to find the Hermite normal form and the Smith normal form of an integral matrix. (3) The preparation of programs using modular arithmetic for the computation of the elementary divisors of an integral matrix.

M. NEWMAN: The Applications of Number Theory to Computation

The purpose of this report is to summarize my Air Force sponsored research on the applications of number theory (in particular modular arithmetic and multiprecision integral arithmetic) to problems of computation, using available serial computers (ILLIAC IV unfortunately no longer exists).

The principal objective of my research is to apply number-theoretic ideas to problems of numerical analysis and computation. In particular, applications of modular and multiprecision arithmetic are being made to the following topics:

(1) The exact solution of an integral system of linear equations, and the exact computation of the determinant of the system;

(2) The determination of the exact inverse of an integral matrix using minimal storage;

(3) The determination of the rank and a basis for the null space of an integral matrix;

(4) The determination of all rational solutions of an integral system of linear equations;

(5) The computation of the Hermite normal form of an integral matrix;

(6) The computation of the Smith normal form of an integral matrix;

(7) The computation of the elementary divisors of an integral matrix.

Programs to perform (1), (2), (3), and (4) using modular

arithmetic have been prepared and checked. The programs are written in a highly portable version of FORTRAN and were implemented without difficulty on a number of different computers, including a microcomputer.

Programs to perform (5), (6), and (7) have been prepared in two versions: the first set of programs uses the standard integer arithmetic provided by the host computer, and are limited by the usual problems of integer overflow. The second set uses multiprecision integer arithmetic subroutines written by the proposer, and are guaranteed to produce the correct result. Under investigation is the problem of whether or not residue arithmetic may be employed in the computation of these important normal forms.

Two Master's Theses on this topic were written under my supervision during this period, by G. Rasmussen and R. Demb.

A report on the work performed is under preparation and will appear in Math. Comp. at some future time.

M. Newman

PUBLICATIONS October 1, 1980 to date

1. Matrices of finite period and some special linear equations, Linear andd Multilinear Algebra 8 (1980), 189-195.

2. On a problem suggested by Olga Taussky-Todd, Illinois J. Math. 24 (1980), 156-158.

3. A surprising determinantal inequality for real matrices (with C.R. Johnson), Math. Ann. 247 (1980), 179-186.

4. A note on cospectral graphs (with C.R. Johnson), J. Comb. Theory B 28 (1980), 96-103.

5. Positive definite matrices and Catalan numbers (with F.T. Leighton), Proc. Amer. Math. Soc. 79 (1980), 177-181.

6. Gersgorin revisited, Linear Algebra and its Applications 30 (1980), 247-249.

7. Determinants of abelian group matrices (with M. Mahoney), Linear and Multilinear Algebra 9, (1980), 121-132.

8. Determinants of circulants of prime power order, Linear and Multilinear Algebra 9 (1980), 187-191.

9. Simple groups of square order and an interesting sequence of primes (with D. Shanks and H.C. Williams), Acta Arith. 38 (1980), 129-140.

10. A radical diophantine equation, to appear in J. Number Theory.

11. Cyclotomic units and Hilbert's Satz 90, to appear in Acta Arith.

12. Pseudo-similarity and partial unit regularity (with F.J. Hall, R.E. Hartwig, and I.J. Katz), to appear in Czech. J. Math.

13. Similarity over SL(n,F), to appear in Linear and Multilinear Algebra.

14. On a problem of H.J. Ryser, to appear in Linear and Multilinear Algebra.

15. Lyapunov revisited, to appear in Linear Algebra and its Applications.

16. A result about determinantal divisors, to appear in Linear and Multilinear Algebra.

Abstract

R.C. Thompson: MATRIX EQUALITIES AND INEQUALITIES

An ongoing program studying singular values and invariant factors has been continued. Recent work includes a partially successful attempt to analyze how the invariant factors of integral matrices behave when matrices add. Nine papers have been submitted to journals (four so far accepted), with two more nearly ready for submission, and several others being prepared. Invitations have been accepted to write the Encyclopedia of Mathematics volume on Matrix Inequalities, and to deliver the keynote lecture on "Core Linear Algebra" at the April 1982 SIAM meeting on Applied Linear Algebra.

R.C. THOMPSON: Matrix Equalities and Inequalities

The proposed research was organized into four compartments. I discuss these in turn.

<u>(1) The matrix exponential</u>. I have conjectured that if H,K are Hermitian matrices, unitary matrices U,V will always exist so that

$$e^{iH}e^{iK} = e^{i(UHU^* + VKV^*)}, \qquad i = \sqrt{-1}\,,\ e = 2.718\ldots$$

* denotes conjugate transpose

where U,V depend on H,K. A renewed attempt was made to prove this conjecture, but success was minimal. Numerous special cases can be settled, and the key concepts that are needed from Lie theory have been identified, but the unravelling of the details remains beyond reach. The author's belief is that several years of sustained effort will be needed to settle this conjecture.

Several manuscripts bearing on this question have been prepared, but none have been submitted to journals.

<u>(2), (3) Matrix valued inequalities.</u> If A is a matrix, let $|A| = (AA^*)^{\frac{1}{2}}$ be its matrix valued absolute value. I proved several years ago that if A,B are matrices, then

$$|A + B| \leq U|A|U^* + V|B|V^*$$

where unitaries U,V depend on A,B. Here $\leq$ means the difference of the two sides is semidefinite. It has now been shown that this inequality is valid if the matrices have quaternion entries. Whether this is so was specifically asked in the proposal, and now has been settled affirmatively.

Several years ago I conjectured that if A is a matrix with U its unitary factor from the polar decomposition, and if V is any other unitary, we always will have

$$|A - U| \leq \tfrac{1}{2} W_1 |A - V| W_1^* + \tfrac{1}{2} W_2 |A - V| W_2^*$$

for appropriate unitaries W , W . That is: U is nearer A than any other unitary V is. This conjecture is true for 2 x 2 matrices, but cannot so far be settled for the 3 x 3 case, in spite of copious quantities of computer work that repeatedly verifies its truth in numerical cases.

Considerable (theoretical) effort has been expended on this conjecture, with not much success. Further work is planned. This is one of the items, not explicitly listed in the grant proposal, but caught under the general theme of "study of matrix valued inequalities".

<u>(4) Harmonic analysis.</u> A matrix question coming from harmonic analysis, previously solved by this author, was written into a paper and submitted to a journal (#4 below).

<u>Other work not mentioned in the grant proposal.</u>

(i) How do the invariant factors of integral matrices behave when the matrices add? A five month investigation resulted in a complete solution for 2 x 2 and 3 x 3 matrices, and a conjectured solution for n x n matrices.

(ii) An invitation has been accepted to write the Matrix Inequalities volume for the Encyclopedia of Mathematics. The Table of Contents for this volume is attached to this report. It will take several years' work to bring this volume to

completion. Work on it was begun under the just expired grant.

(iii) An invitation has been accepted to deliver one of the principal lectures at the SIAM Applied Linear Algebra meeting, April, 1982. This meeting has six principal speakers, five on various aspects of applied linear algebra, plus (to keep the meeting honest) one on core linear algebra. I am the invited speaker on core linear algebra, and work on it was begun under the just expired grant.

R.C. Thompson

PUBLICATIONS October 1, 1980 to date

1. P-adic matrix valued inequalities, submitted.

2. The Jacobi-Gundelfinger-Frobenius-Iohvidov rule and the Hasse symbol, to appear in Letters in Linear Algebra.

3. The Smith form, the inversion rule for 2 x 2 matrices, and the uniqueness of the invariant factors for finitely generated modules, to appear in Letters in Linear Algebra.

4. Doubly stochastic, unitary, unimodular, and complex orthogonal power embeddings, submitted.

5. A converse to the Ostrowski-Taussky determinantal inequality, to appear in Linear Algebra and Applications.

6. Cyclic relations and the Goldberg coefficients in the Campbell-Baker-Hausdorff formula, submitted.

7. An inequality for invariant factors, submitted.

8. The matrix valued triangle inequality - quaternion version, to appear in Linear and Multilinear Algebra.

9. The true growth rate and the inflation balancing principle, submitted.

The following two manuscripts are essentially complete and about to be sent to journals.

10. Left multiples and right divisors of integral matrices.

11. Sums of integral matrices.

Some of these papers represent work from earlier AFOSR grants, some from this grant alone.

Seven manuscripts are in various stages of preparation, and will be listed in future grants as they eventually get to journals.

A Beginner's Survey of Matrix Inequalities

by R.C. Thompson

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1. Dominance and Lie theory
2. Inequalities for matrices with continuous entries
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Abstract

H. Minc: VAN DER WAERDEN PERMANENT CONJECTURE.
THEORY OF PERMANENTS AND ITS APPLICATIONS.
INEQUALITIES.

Research during the period included:

(1) Study of Egoryčev's proof of the van der Waerden conjecture. The proof was reformulated and somewhat extended. Alternative proofs, different in parts from Egoryčev's proof, were devised.

(2) The developments in the theory of permanents in the last four years were surveyed and discussed. Present status of 30 conjectures and 13 problems was discussed. A comprehensive bibliography was compiled.

(3) New rearrangement inequalities for products and sums of nonnegative numbers were obtained.

H. MINC: (I) van der Waerden Permanent Conjecture

The most famous and one of the most important problems in the theory of permanents is the so-called van der Waerden conjecture: if A is a doubly stochastic n x n matrix, then

$$\operatorname{per}(A) \geqq n!/n ,$$

where equality can hold if and only if all the entries of A are 1/n. The problem actually originated in a theorem of König (1916) who proved that the permanent of a doubly stochastic matrix is always positive. For over half a century the conjecture resisted all attempts of solution, until late in 1980 when Egoryčev proved it completely. One of the effects of this historical proof was that much of the research in progress on permanents of doubly stochastic matrices, including applications to the dimer problem, enumerations of various combinatorial problems, etc., had to be reevaluated.

In [5] a version of Egoryčev's proof is given in which some of the auxiliary results are somewhat generalized. In [4] it is shown that the concluding argument in Egoryčev's proof can be replaced by either of two alternative proofs.

(II) Theory of Permanents and Its Applications

The last four years witnessed an increased research activity in the theory of permanents and its applications. At least 75 new research papers on the subject and similar publications were written or were actually published during this four-year period; this represents nearly 20% of the total literature on permanents in the last 170 years.

In [5] the developments in the area are surveyed. The

paper (56 typed pages) includes a detailed discussion of Egoryčev's proof of the van der Waerden conjecture (see above), a section on bounds of permanents of (0,1)-matrices, and on their applications to the dimer problem, to the problem of enumeration of Latin rectangles, and other topics. The paper contains a report on the current status of each of the 20 conjectures and 10 problems, which were listed as unsolved in Minc's "Permanents" (1978). Additional 10 conjectures and 3 problems are proposed. An extensive, virtually complete, bibliography is appended, each item with a short abstract. Finally, 30 additional references (with abstract) are listed for the period preceding 1978; these failed to be included in the bibliography of "Permanents".

(III) Inequalities

Let $(a_{1j}, a_{2j}, \ldots, a_{m_j j})$ be non-negative m_j-tuples, $j = 1, \ldots, n$, where $m_1 \geqq \ldots \geqq m_n \geqq 1$, and $\sum_{j=1}^{n} m_j = M$. Let

$$\alpha'_{11} \leqq \alpha'_{21} \leqq \ldots \leqq \alpha'_{m_1 1} \leqq \alpha'_{12} \leqq \alpha'_{22} \leqq \ldots \leqq \alpha'_{m_2 2} \leqq \ldots \leqq \alpha'_{1n} \leqq \alpha'_{2n} \leqq \ldots \leqq \alpha'_{m_n n}$$

be the M numbers a_{ij}, $i = 1, \ldots, m_j$, $j = 1, \ldots, n$, arranged in non-decreasing order, and

$$\alpha^*_{11} \geqq \alpha^*_{21} \geqq \ldots \geqq \alpha^*_{m_1 1} \geqq \alpha^*_{12} \geqq \alpha^*_{22} \geqq \ldots \geqq \alpha^*_{m_2 2} \geqq \ldots \geqq \alpha^*_{1n} \geqq \alpha^*_{2n} \geqq \ldots \geqq \alpha^*_{m_n n}$$

be the same numbers arranged in non-increasing order.

In [1] we prove inequalities of the following types. If $a_{ij} \leq 1$, $i = 1, \ldots, m_j$, $j = 1, \ldots, n$, then

$$\sum_{j=1}^{n} \prod_{i=1}^{m_j} a_{ij} \leqq \sum_{j=1}^{n} \prod_{i=1}^{m_j} \alpha'_{ij}.$$

and

$$\prod_{j=1}^{n} (1 + a_{1j} a_{2j} \ldots a_{m_j j}) \leqq \prod_{j=1}^{n} (1 + \alpha'_{1j} \alpha'_{2j} \ldots \alpha'_{m_j j}).$$

If $a_{ij} \geq 1$, $i = 1, \ldots m_j$, $j = 1, \ldots, n$, then

$$\sum_{j=1}^{n} \prod_{i=1}^{m_j} a_{ij} \leqq \sum_{j=1}^{n} \prod_{i=1}^{m_j} \alpha_{ij}^*$$

and

$$\prod_{j=1}^{n} (1 + a_{1j}a_{2j} \ldots a_{m_j j}) \leqq \prod_{j=1}^{n} (1 + \alpha_{1j}^* \alpha_{2j}^* \ldots \alpha_{m_j j}^*).$$

In either case

$$\prod_{j=1}^{n} \sum_{i=1}^{m_j} a_{ij} \geqq \prod_{j=1}^{n} \sum_{i=1}^{m_j} \alpha_{ij}^*.$$

Some of these inequalities were used to establish bounds for permaments of nonnegative matrices.

H. Minc

PUBLICATIONS October 1, 1980 to date

1. Rearrangement inequalities, Proc. Roy. Soc. Edinburgh 86A (1980), 339-345.

2. Inverse elementary divisor problem for doubly stochastic matrices, Linear and Multilinear Algebra (to appear).

3. Inverse elementary divisor problem for nonnegative matrices, Proc. Amer. Math. Soc. (to appear).

4. A note on Egoryčev's proof of the van der Waerden conjecture, Linear and Multilinear Algebra (to appear).

5. The theory of permanents 1878-1981, (submitted).

E.C. JOHNSEN: Elementary Factorizations of Stochastic and Doubly Stochastic Matrices

E.C. Johnsen has been investigating the questions of (1) when stochastic matrices factor into elementary stochastic matrices and (2) when doubly stochastic matrices factor into elementary doubly stochastic matrices. Results obtained earlier on problem (1) are being reworked and rewritten.* These results give rather general sufficient conditions for the elementary factorization of stochastic matrices. Problem (2) appears to be considerably more difficult than problem (1). Partial affirmative results on problem (2) have been obtained by the author for the case of orthostochastic matrices of order 3. It is known that not every doubly stochastic matrix of order $n \geq 3$ and that not every orthostochastic matrix of order $n \geq 4$ (M. Marcus) can be factored into elementary doubly stochastic matrices. Thus the general factorization existence question is still open for orthostochastic matrices of order 3.

These problems are of applied interest because of the direct relation between elementary factorizations of stochastic and doubly stochastic matrices and sequential allocation and reallocation programs for multiple commodities at fixed sources and destinations (blending problems).

* Johnsen, E.C., "Stochastic Matrices II. Some Factorizations into Elementary Stochastic Matrices"

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